Weak Convergence of Laws on $\mathbb{R}^K$ with Common Marginals

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Abstract

We present a result on topologically equivalent integral metrics (Rachev, 1991, Müller, 1997) that metrize weak convergence of laws with common marginals. This result is relevant for applications, as shown in a few simple examples.

Key Words: Copula, Bounded Lipschitz Metric, Probability Metric, Weak Convergence.

Classification AMS: 60B10, 60E05.

1 Introduction

In applications, it is often necessary to resort to approximations in the study of the properties of stochastic models. The original process can be replaced by a simpler one whose characteristics are already known or easier to study. This requires some stability of the model, which is usually represented in terms of integral probability metrics (Rachev, 1991, Müller, 1997). Among the many possible situations, a typical one is the difficulty to deal with the dependence properties of stochastic processes. For example, the study of the central limit problem for the standardised partial sum of dependent random variables requires the introduction of dependence conditions (mixing conditions, e.g. Doukhan, 1994, or weak dependence conditions, e.g. Doukhan and Louhichi, 1999). These dependence conditions are used to relate past and future realisations

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of some stochastic process with some "independent version": an example in the text, below, will make this statement less elusive. In this case as in many other applications, the univariate marginal distributions of the approximation are the same as the original ones. When the marginal distributions of the original vector and the approximating one are the same, there are some results pertaining weak convergence metrized by integral probability metrics that have not been explored before. The goal of this paper is to relate the minimal generator of that metrizes weak convergence under an integral probability metric to more general classes of test functions. As a few examples show, this is particularly important when dealing with convergence rates of moments between the original process and the approximation. To give a simple illustration of the applications to be discussed below, consider $X_n := (X_{1n}, X_{2n})$ converging weakly to $X := (X_1, X_2) \in \mathbb{R}^2$. This implies that for any continuous bounded function, say f, $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$. If $f(X)$ is only uniformly integrable, the rate of convergence is not the one obtained under the Dudley metric (bounded Lipschitz metric), as we need to truncate. However, if $law(X_{in}) = law(X_i)$ $(i = 1, 2)$, under some conditions on f, the results of this paper show that truncation is not necessary and a tighter bound can be achieved. In particular, f may even be discontinuous for the convergence of moments of f to be satisfied and the marginal laws can be arbitrary as long as they are in common. No structure is imposed on $law(X_n)$ and $law(X)$ apart from the marginals being in common and the assumed weak convergence.

Below we give some background information also to set up the notation (Section 2). Then, the result of the paper is stated (Section 3) followed by some applications (Section 4). The proofs are in Sections 5 and 6.

2 Background Material and Notation

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are two laws. For a class of functions $\mathfrak{F}$ we define the integral metric

$$d_{\mathfrak{F}}(\mathbb{P}, \mathbb{Q}) := \sup_{f \in \mathfrak{F}} \left| \int f(x)\, d(\mathbb{P} - \mathbb{Q})(x) \right|. \tag{1}$$

These metrics have been used by Rachev (1991) to study the stability of stochastic systems as well as by Müller (1997). Suppose f is a function with domain in the metric space $(\mathcal{X}, r)$. Define the supremum norm $\|f\|_\infty := \sup_{x \in \mathcal{X}} |f(x)|$ and the Lipschitz norm as $\|f\|_L := \sup_{x,y \in \mathcal{X}} |f(x) - f(y)| / r(x, y)$. Define the bounded Lipschitz metric of f as the semimetric $\|f\|_{BL} := \|f\|_\infty + \|f\|_L$. We say that

a function f belongs to the class $BL_1 := BL_1(\mathcal{X})$ if $\|f\|_{BL} \leq 1$. The metric space $(\mathcal{X}, r)$ is implicit in this definition.

When $\mathfrak{F} := BL_1$, and $(\mathbb{P}_n)_{n\in\mathbb{N}}$ $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ are two sequences of laws defined on $(\mathcal{X}, r)$, (1) can be used to metrize weak convergence of $\mathbb{P}_n$ and $\mathbb{Q}_n$ to each other (and often, $\mathbb{Q}_n = \mathbb{Q}$ is fixed and understood as the limiting value of $(\mathbb{P}_n)_{n\in\mathbb{N}\cup\{\infty\}}$). Suppose $\mathcal{X} \subseteq \mathbb{R}^K$ and $\mathbb{P}_n$ and $\mathbb{Q}_n$ are the laws of $X_n := (X_{n1}, ..., X_{nK})$ and $X'_n := (X'_{n1}, ..., X'_{nK})$ such that $law(X_{nk}) = law(X'_{nk})$ so that $\mathbb{P}_n$ and $\mathbb{Q}_n$ have the same univariate marginals. This paper gives a result about the generator $\mathfrak{F}$ of (1) when $\mathbb{P}_n$ and $\mathbb{Q}_n$ satisfy the just mentioned restriction. The space $\mathcal{X}$ will be equipped with the l_p distance denoted by $r_p(x,y) := \left(\sum_{k=1}^K |x_k - y_k|^p\right)^{1/p}$ ($p \in [1, \infty]$, with the obvious modification for $p = \infty$). Finally, $M_1 := M_1(\mathcal{X})$ is the class of coordinatewise nondecreasing functions from $\mathcal{X}$ to $[0,1]$.

3 Statement of Result

When the marginals between two laws are in common, in the sense defined above, we have the following relationship between the generators BL_1 and M_1.

Theorem 1 *Suppose $(\mathbb{P}_n)_{n\in\mathbb{N}}$ and $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ have common marginals. Then, $d_{BL_1}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$ if and only if $d_{M_1}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$.*

Remark 2 *Surprisingly, the two metrics d_{BL_1} and d_{M_1} are topological equivalent when $\mathbb{P}_n$ and $\mathbb{Q}_n$ have same marginals. Since the indicator of $[x, \infty) := [x_1, \infty) \times \cdots \times [x_K, \infty)$ is coordinatewise increasing,*

$$\sup_{x\in\mathcal{X}} |\mathbb{P}_n([x,\infty)) - \mathbb{Q}_n([x,\infty))| \leq d_{M_1}(\mathbb{P}_n, \mathbb{Q}_n),$$

so that weak convergence of sequences with common marginals is equivalent to the above uniform convergence. By the Portmanteau Theorem, we know that this is not true in general.

In applications it is important to use functions that are not necessarily continuous. However, for actual calculations, it is much easier to compute d_{BL_1}. We give a simple illustrative example.

Example 3 *Suppose*

$$\begin{aligned} Y_n &= \sum_{s=0}^{\infty} a_s Z_{n-s} \\ Y_n' &= \sum_{s=0}^{n-1} a_s Z_{n-s} + \sum_{s=n}^{\infty} a_s Z_{n-s}', \end{aligned}$$

where $(Z_t)_{t\in\mathbb{Z}}$ *are iid integrable random variables and* $(Z_t')_{t\in\mathbb{Z}}$ *is an independent copy of* $(Z_t)_{t\in\mathbb{Z}}$*. Hence* Y_n' *is independent of* Y_0*. Consider the laws of* $X_n := (Y_0, Y_n)$ *and* $X_n' := (Y_0, Y_n')$*, say* $\mathbb{P}_n$ *and* $\mathbb{Q}_n$*, where* X_n *and* X_n' *take values in* $\left(\mathbb{R}^2, r_1\right)$*. We have*

$$d_{BL_1}(\mathbb{P}_n, \mathbb{Q}_n) \le \mathbb{E} r_1 (X_n, X_n') = \mathbb{E}\,|Y_n - Y_n'| \le 2\mathbb{E}\,|Z_0| \sum_{s=n}^{\infty} |a_s|\,,$$

by the Lipschitz condition. If $\sum_{s=n}^{\infty} |a_s| \to 0$ *as* $n \to \infty$*, Theorem 1 gives* $d_{M_1}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$ *as well. In words, this means that the law of* X_n *converges to the product of the marginals (*X_n' *has independent components) and the convergence holds for the expectation of more general classes of functions than* BL_1 *(e.g. Remark 2).*

As the previous example shows, it is also important to relate the two metrics in order to derive rates of convergence.

Theorem 4 *Suppose* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *and* $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ *are as in Theorem 1 and recall that* $\mathbb{P}_n$ *and* $\mathbb{Q}_n$ *are defined on* $\left(\mathcal{X} \subseteq \mathbb{R}^K, r_p\right)$*. Then,*

$$d_{M_1}(\mathbb{P}_n, \mathbb{Q}_n) \le \min\left\{2^{3/2} K^{(p-1)/(2p)} d_{BL_1}(\mathbb{P}_n, \mathbb{Q}_n)^{1/2}, 1\right\}.$$

4 Two Applications

We give two applications. In both cases, we show that relating the integral probability metric of the class BL_1 to the class M_1 allows us to derive upper bounds for integral probability metrics based on unbounded classes of functions, not necessarily continuous. In particular we shall define $MRC_+ := MRC_+(\mathbb{R})$ to be the class of positive nondecreasing functions on $\mathbb{R}$, continuous from the right.

4.1 Integral Probability Metrics for Products of Unbounded Positive Functions

Suppose

$$\mathfrak{G} := \left\{ \prod_{k=1}^{K} g_k : g_k \in MRC_+, k = 1, ..., K \right\}$$

is a class of functions such that each element g is the pointwise product of positive nondecreasing functions continuous from the right. Even if $g \in \mathfrak{G}$ is unbounded, the integral probability metric with respect to $\mathfrak{G}$ can be bounded in terms of $M_1\left(\mathbb{R}^K\right)$ and $BL_1\left(\mathbb{R}^K\right)$ by Theorem 4.

Corollary 5 *Suppose $\mathbb{P}$, $\mathbb{Q}$ are the laws of X, $X' \in \mathbb{R}^K$ having same univariate marginals. Set $\theta := d_{M_1(\mathbb{R}^K)}(\mathbb{P},\mathbb{Q})/2$, then*

$$d_{\mathfrak{G}}(\mathbb{P},\mathbb{Q}) \leq \int_0^{\theta} \prod_{k=1}^{K} Q_{g_k}(s)\, ds,$$

where $Q_{g_k}(s) = \inf\{x : \Pr(g_k(X_k) > x) \leq s\}$.

Remark 6 *By simple approximating arguments, we can extend Corollary 5 from $\mathfrak{G}$ to the convex hull of $\mathfrak{G} \cup (-\mathfrak{G})$, where $g \in (-\mathfrak{G})$ if $-g \in \mathfrak{G}$. Details are left to the reader.*

4.2 Covariance Inequalities

Suppose, Y and Z are two random variables. Define

$$\alpha := \alpha(Y,Z) := 2 \sup_{z,y\in\mathbb{R}} \left|\Pr(Y > y, Z > z) - \Pr(Y > y)\Pr(Z > z)\right|, \tag{2}$$

which is a simple version of the strong mixing coefficient

$$\alpha(\mathcal{A},\mathcal{B}) := 2\sup\{Cov(I_A, I_B) : A \in \mathcal{A}, B \in \mathcal{B}\}, \tag{3}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two sigma algebras. Strong mixing is a commonly used dependence condition and limit theorem for dependent random variables are often proved using it. The coefficient α is the basis of Rio's covariance inequality (e.g. Rio 2000, Theorem 1.1), which is known to be sharp. Using the results of this paper we can prove the following extension of the just mentioned inequality.

Corollary 7 *Suppose (Y,Z) is a random vector with values in $\left(\mathbb{R}^2, r_1\right)$. Set*

$$\theta := 2\min\left\{\left|8 d_{BL_1}\left(\mathbb{P}_{(Y,Z)}, \mathbb{P}_Y\mathbb{P}_Z\right)\right|^{1/2}, 1\right\},$$

where $\mathbb{P}_{(Y,Z)}$ is the law of (Y, Z) and $\mathbb{P}_Y\mathbb{P}_Z$ it the product of the marginals. Suppose $g_Y, g_Z \in MRC_+$. Then,

$$Cov\left(g_Y\left(Y\right), g_Z\left(Z\right)\right) \leq 2\int_0^{\theta} Q_{g_Y}\left(s\right) Q_{g_Z}\left(s\right) ds,$$

where $Q_{g_Y}\left(s\right) := \inf\left\{y : \Pr\left(\left|g_Y\left(Y\right)\right| > y\right) \leq s\right\}$ and $Q_{g_Z}\left(s\right) := \inf\left\{z : \Pr\left(\left|g_Z\left(Z\right)\right| > z\right) \leq s\right\}$.

Remark 8 *To show a similar result, Doukhan and Louhichi (1999, Lemma 1) require the marginals to be Lipschitz continuous. Theorem 4 shows that Lipschitz continuity is not necessary. In general the strong mixing condition is satisfied if $d_{BL_1} \to 0$, and we restrict A and B in (3) to be half open intervals, as in (2).*

5 Weak Convergence with Common Marginals

Suppose $X := (X_1, ..., X_K) \in \mathcal{X} \subseteq \mathbb{R}^K$ is a vector of random variables with law $\mathbb{P}$ and $P_k := law\left(X_k\right)$ $(k = 1, ..., K)$. Then, there exists a function $C : [0, 1]^K \to [0, 1]$ such that

$$\Pr\left(X_1 \in A_1, ..., X_K \in A_K\right) = C\left(P_1\left(A_1\right), ..., P_K\left(A_K\right)\right) \tag{4}$$

for $A_k = [-\infty, x_k] \subset \mathbb{R}$, $(k = 1, ..., K)$. The function C is called copula function and the above representation holds even for sets more general than A_k (Scarsini, 1989). Suppose C' is a copula distinct from C above. The distance between these two copulae can be measured in terms of an integral probability metric generated by a class of functions $\mathfrak{F}$, i.e. $d_{\mathfrak{F}}\left(C, C'\right)$. Define

$$\tilde{F}_k\left(x, v\right) := \Pr\left(X_k < x\right) + v \Pr\left(X_k = x\right),\ v \in [0, 1] \text{ and } x \in \mathbb{R}.$$

Define the operator $\mathcal{Q}^* : \mathbb{R}^K \to [0, 1]^K$ such that

$$\mathcal{Q}^* X := \left(\tilde{F}_1\left(X_1, V_1\right), ..., \tilde{F}_K\left(X_K, V_K\right)\right),$$

where $(V_1, ..., V_K)$ are iid $[0, 1]$ uniform random variables independent of X. Then, $U := \mathcal{Q}^* X$ is a random vector with uniform $[0, 1]$ marginals (Rüschendorf and de Valk, 1993, Proposition 1). If P_k $(k = 1, ..., K)$ is absolutely continuous with respect to the Lebesgue measure, from (4) we deduce U has law C. Define also the operator $\mathcal{Q} : [0, 1]^K \to \mathbb{R}^K$ such that for some $[0, 1]^K$ uniform random vector U with law C,

$$\mathcal{Q}U := \left(P_1^{-1}\left(U_1\right), ..., P_K^{-1}\left(U_K\right)\right),\ \text{(where } P_k^{-1}\left(u\right) := \inf\left\{x : \Pr\left(X \leq x\right) \geq u\right\}\text{)}$$

so that $\mathcal{Q}U \stackrel{d}{=} X$ ($\stackrel{d}{=}$ is equality in distribution). Hence, if the marginals of $\mathcal{Q}U$ are not continuous, the copula is not unique (e.g. Scarsini, 1989). Here, the term copula will refer to the law of $U \stackrel{d}{=} \mathcal{Q}^* X$, which is unique.

Denote $M_1\left([0,1]^K\right)$ to be the class of functions M_1, but with support in $[0,1]^K$; similarly, define $BL_1\left([0,1]^K\right)$. Moreover define

$$\mathfrak{B} := \left\{g : [0,1]^K \to \mathbb{R} : g := f \circ \mathcal{Q}, f \in BL_1(\mathcal{X})\right\}$$

so that $\mathfrak{B}$ is the class of functions obtained from functions in $BL_1(\mathcal{X})$ using the composition of f with the transformation $\mathcal{Q}$.

Lemma 9 *Suppose X and X' are random variables with values in $\mathcal{X} \subseteq \mathbb{R}^K$ and laws $\mathbb{P}$ and $\mathbb{Q}$ having common marginals and with copula C and C' respectively. Then*

$$d_{M_1(\mathcal{X})}(\mathbb{P}, \mathbb{Q}) = d_{M_1\left([0,1]^K\right)}(C, C')$$

and

$$d_{BL_1(\mathcal{X})}(\mathbb{P}, \mathbb{Q}) = d_{\mathfrak{B}}(C, C') \geq d_{BL_1\left([0,1]^K\right)}(C, C').$$

Proof. Note that if $f \in M_1(\mathcal{X})$ then $f \circ \mathcal{Q} \in M_1\left([0,1]^K\right)$ and if $g \in M_1\left([0,1]^K\right)$ then $g \circ \mathcal{Q}^{-1} \in M_1(\mathcal{X})$, where $\mathcal{Q}^{-1}X := (P_1(X_1), ..., P_K(X_K))$ and $P_k(x) := P_k([-\infty, x])$. This follows because $\mathcal{Q}$ is nondecreasing. Since $\mathcal{Q}$ is a measurable map (if we consider the sigma algebras generated by Borel sets of $\mathcal{X}$ and $[0,1]^K$) by a change of variables,

$$d_{M_1(\mathcal{X})}(\mathbb{P}, \mathbb{Q}) = \sup_{f \in M_1(\mathcal{X})} \left| \int f d(\mathbb{P} - \mathbb{Q}) \right| = \sup_{f \in M_1(\mathcal{X})} \left| \int f \circ \mathcal{Q} d(C - C') \right| = d_{M_1\left([0,1]^K\right)}(C, C').$$

The last equality follows because $f = g \circ \mathcal{Q}^{-1} \in M_1(\mathcal{X})$ (for some $g \in M_1\left([0,1]^K\right)$), as mentioned above. The second part of the lemma follows by change of variables. We only need to note that functions in $\mathfrak{B}$ are bounded, but not necessarily Lipschitz unless the marginals of X are Lipschitz continuos. Hence, $BL_1\left([0,1]^K\right) \subseteq \mathfrak{B}$. ■

For $U \in [0,1]^K$ define $\bar{U} = (1 - U_1, ..., 1 - U_K)$ and similarly for $u \in [0,1]^K$ define $\bar{u}$. Then, $\hat{C}(u) := \Pr\left(\bar{U} \leq u\right)$ is the copula of $\bar{U}$ so that $\bar{C}(u) := \hat{C}(\bar{u}) = \Pr(U \geq u)$ is the survival copula (vector inequalities are meant to hold elementwise). We recall that the copula is Lipschitz of order and constant one with respect to the l_1 norm,

$$|C(u_1, ..., u_K) - C(v_1, ..., v_K)| \leq r_1(u, v) = \sum_{k=1}^{K} |u_k - v_k|. \tag{5}$$

Since $\hat{C}(u)$ is a copula, (5) also implies

$$\left|\bar{C}(u)-\bar{C}(v)\right|=\left|\hat{C}(\bar{u})-\hat{C}(\bar{v})\right|\leq r_1(\bar{u},\bar{v})=r_1(u,v). \tag{6}$$

Hence, we have the following.

Lemma 10

$$\sup_{f\in M_1\left([0,1]^K\right)}\left|\int fd\left(C-C'\right)\right|=\sup_{u\in[0,1]^K}\left|\bar{C}(u)-\bar{C}'(u)\right|.$$

Proof. For simplicity, suppose $f(0_K)=0$ (0_K is the K dimensional vector of zeros). Define

$$f_m:=\sum_{i\in\mathcal{I}_m}\Delta_m f_i I\{u\in[u_i,1]\},$$

where $[u_i,1]:=[u_{i_1},1]\times\cdots\times[u_{i_K},1]\subset[0,1]^K$, $u_{i_k}\leq u_{i_k+1}$, and $\Delta_m f_i\geq 0$ such that $\sum_{i\in\mathcal{I}_m}\Delta_m f_i\leq 1$, $i\in\mathcal{I}_m\subset\mathbb{N}^K$ and $\#\mathcal{I}_m=2^m$ ($I\{...\}$ is the indicator function). Choosing $(u_i)_{i\in\mathcal{I}_m}$ such that $\Delta_m f_i\leq c2^{-m}$ (for some bounded absolute constant c), $\mathbb{E}|f_m-f|\to 0$ as $m\to\infty$ for any $f\in M_1$. Such a construction and its convergence can be shown by simple manipulation of the usual approximation of functions in terms of indicator functions (e.g. see the proof of Theorem 13.5 in Billingsley, 1995) together with monotonicity. Then,

$$\begin{aligned}\left|\int f_m d\left(C-C'\right)\right| &= \left|\sum_{i\in\mathcal{I}_m}\Delta_m f_i\int I\{u\in[u_i,1]\}\,d\left(C-C'\right)(u)\right|\\ &\leq \sum_{i\in\mathcal{I}_m}\Delta_m f_i\max_{i\in\mathcal{I}_m}\left|\int I\{u\in[u_i,1]\}\,d\left(C-C'\right)(u)\right|.\end{aligned}$$

By continuity of C (i.e. (5)), the above display is unaffected if $[u_i,1]$ in the definition of f_m does not include the left boundary in any/some of the coordinates. Hence, we can regard $\sup_m f_m$ to be a nondecreasing function (not necessarily right continuous). Hence, taking sup with respect to m and f, we deduce

$$\sup_{f\in M_1}\left|\int fd\left(C-C'\right)\right|\leq\sup_{u\in[0,1]^K}\left|\Pr(U\geq u)-\Pr(U'\geq u)\right|=\sup_{u\in[0,1]^K}\left|\bar{C}(u)-\bar{C}'(u)\right|,$$

where U and U' are uniform $[0,1]^K$ random vectors with law C and C' respectively. The lower bound follows noting that the indicator of $[u_i,1]\subset[0,1]^K$ is coordinatewise increasing, where the same remark about the boundaries applies. ■

Proof of Theorem 1. Suppose $d_{BL_1(\mathcal{X})}(\mathbb{P}_n,\mathbb{Q}_n)\to 0$. By Lemma 9, $d_{BL_1\left([0,1]^K\right)}(C_n,C'_n)\to 0$ where C_n and C'_n are the copulae corresponding to $\mathbb{P}_n$ and $\mathbb{Q}_n$. Then, by (6) (i.e. continuity),

$$\sup_{u\in[0,1]^K}\left|\bar{C}_n(u)-\bar{C}'_n(u)\right|\to 0 \tag{7}$$

and $d_{M_1\left([0,1]^K\right)}(C_n, C'_n) \to 0$ by Lemma 10. Finally, Lemma 9 implies $d_{M_1(\mathcal{X})}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$.

Now suppose $d_{M_1(\mathcal{X})}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$. Since the class of indicators of right half spaces are in $M_1(\mathcal{X})$,

$$\sup_{x\in\mathcal{X}} |\Pr(X_n \geq x) - \Pr(X'_n \geq x)| \leq d_{M_1(\mathcal{X})}(\mathbb{P}, \mathbb{Q}) \to 0,$$

and we must have $d_{BL_1(\mathcal{X})}(\mathbb{P}_n, \mathbb{Q}_n) \to 0$, by weak convergence (because in $\mathcal{X} \subseteq \mathbb{R}^K$ the sets $[x, \infty] := [x_1, \infty] \times \cdots \times [x_K, \infty] \subset \mathbb{R}^K$ are a separating class). ■

Proof of Theorem 4. At first we show that the bound holds for C_n and C'_n. Let $f(u) := I\{u \in [v, 1]\}$ and $g_\epsilon(u) := [1 - r_p([v, 1], u)/\epsilon] \vee 0$, $u \in [0, 1]^K$ ($[v, 1] := [v_1, 1] \times \cdots \times [v_K, 1] \subset [0, 1]^K$). Hence, g_ϵ is a Lipschitz approximation of f (as $\epsilon \to 0$) and such that $\|g_\epsilon\|_{BL} \leq \left(1 + \epsilon^{-1}\right)$. Noting that $g_\epsilon \geq f$, consider the following bound, where dependence on the argument u is suppressed,

$$\begin{aligned}
\int f d(C_n - C'_n) &\leq \int g_\epsilon dC_n - \int f dC'_n = \int g_\epsilon d(C_n - C'_n) + \int (g_\epsilon - f) dC'_n \\
&\leq \left(1 + \epsilon^{-1}\right) d_{BL_1\left([0,1]^K\right)}(C_n, C'_n) + \int (g_\epsilon - f) dC'_n \\
&\qquad \text{[by the previous remarks about } g_\epsilon] \\
&\leq \left(1 + \epsilon^{-1}\right) d_{BL_1\left([0,1]^K\right)}(C_n, C'_n) + r_1(u', v), \qquad (8)
\end{aligned}$$

for some $u' \in [0, 1]^K$ using (6). By construction, u' and v are such that

$$r_p(v, u') = \left(\sum_{k=1}^K |u'_k - v_k|^p\right)^{1/p} \leq \epsilon,$$

implying, by the relations between means (i.e. $r_1(u, v) K^{-1} \leq r_p(u, v) K^{-1/p}$),

$$r_1(u', v) \leq K^{1-1/p} r_p(u', v) \leq K^{(p-1)/p}\epsilon.$$

Substitute this bound in (8). Noting that $\epsilon \leq 1$, choosing ϵ to equate the two terms gives

$$d_{M_1\left([0,1]^K\right)}(C_n, C'_n) \leq 2^{3/2} K^{(p-1)/(2p)} d_{BL_1\left([0,1]^K\right)}(C_n, C'_n)^{1/2},$$

where we have taken sup with respect to f on both sides (f in the class of indicators of right half open intervals $[v, 1] \subset [0, 1]^K$ as defined above) and we have used Lemma 10. Then, by Lemma 9, the previous display implies

$$\begin{aligned}
d_{M_1(\mathcal{X})}(\mathbb{P}_n, \mathbb{Q}_n) &= d_{M_1\left([0,1]^K\right)}(C_n, C'_n) \leq 2^{3/2} K^{(p-1)/(2p)} d_{BL_1\left([0,1]^K\right)}(C_n, C'_n)^{1/2} \\
&\leq 2^{3/2} K^{(p-1)/(2p)} d_{BL_1(\mathcal{X})}(\mathbb{P}, \mathbb{Q})^{1/2}.
\end{aligned}$$

By Lemma 10, we deduce $d_{M_1\left([0,1]^K\right)}\left(C_n, C_n'\right) \leq 1$ and the result follows. ■

6 Proof of Corollaries

Proof of Corollary 5. For any positive function $g_k : \mathbb{R} \to \mathbb{R}$

$$g_k\left(x\right) = \int_0^\infty I\left\{g_k\left(x\right) > s\right\} ds.$$

Hence, suppressing the argument x for ease of notation,

$$\begin{aligned}
\mathrm{I} \quad : \quad &= \int g_1 \cdots g_K d\left(\mathbb{P} - \mathbb{Q}\right) \\
&= \int \prod_{k=1}^K \int_0^\infty I\left\{g_k > s_k\right\} ds_k d\left(\mathbb{P} - \mathbb{Q}\right) \\
&= \int_{\mathbb{R}^K} ds_1 \cdots ds_K \int \prod_{k=1}^K I\left\{g_k > s_k\right\} d\left(\mathbb{P} - \mathbb{Q}\right)
\end{aligned}$$

by Fubini's Theorem. Then,

$$\begin{aligned}
\mathrm{II} \quad : \quad &= \int \prod_{k=1}^K I\left\{g_k > s_k\right\} d\left(\mathbb{P} - \mathbb{Q}\right) \\
&\leq \left|\Pr\left(g_k\left(X_k\right) > s_k, k = 1, .., K\right) - \Pr\left(g_k\left(X_k'\right) > s_k, k = 1, .., K\right)\right| \\
&= \left|\Pr\left(X_k > g_k^{-1}\left(s_k\right), k = 1, .., K\right) - \Pr\left(X_k' > g_k^{-1}\left(s_k\right), k = 1, .., K\right)\right| \\
&\quad \text{[because } g_k \text{ is nondecreasing and right continuous]} \\
&\leq d_{M_1\left(\mathbb{R}^K\right)}\left(\mathbb{P}, \mathbb{Q}\right) = 2\theta,
\end{aligned}$$

and

$$\int \prod_{k=1}^K I\left\{g_k > s_k\right\} d\left(\mathbb{P} - \mathbb{Q}\right) \leq 2 \min\left(\Pr\left(g_k\left(X_k\right) > s_1\right), ..., \Pr\left(g_K\left(X_K\right) > s_K\right)\right)$$

because of the following reasons: the marginals of $\mathbb{P}$ and $\mathbb{Q}$ are the same, and

$$\begin{aligned}
\int \prod_{k=1}^K I\left\{g_k > s_k\right\} d\mathbb{P} \quad &= \quad \int I\left\{g_l > s_l\right\} \prod_{\substack{k=1 \\ k \neq l}}^K I\left\{g_k > s_k\right\} d\mathbb{P} \\
&\leq \quad \min\left(\int I\left\{g_1 > s_1\right\} d\mathbb{P}, ..., \int I\left\{g_K > s_K\right\} d\mathbb{P}\right) \\
&\quad \quad \text{[because the terms in the product take values in } [0,1]\text{]} \\
&= \quad \min\left(\Pr\left(g_1\left(X_1\right) > s_1\right), ..., \Pr\left(g_K\left(X_K\right) > s_K\right)\right).
\end{aligned}$$

From the previous displays,

$$\begin{aligned} \mathrm{II} &\leq 2\min\left(\theta, \Pr\left(g_1\left(X_1\right) > s_1\right), ..., \Pr\left(g_K\left(X_K\right) > s_K\right)\right) \\ &= 2\int_0^\theta \prod_{k=1}^K I\left\{s < \Pr\left(g_k\left(X_k\right) > s_k\right)\right\} ds = 2\int_0^\theta \prod_{k=1}^K I\left\{Q_{g_k}\left(s\right) > s_k\right\} ds, \end{aligned}$$

which implies

$$\mathrm{I} \leq \int_{\mathbb{R}^K} \mathrm{II} ds_1 \cdots ds_K \leq \int_0^\theta \prod_{k=1}^K Q_{g_k}\left(s\right) ds,$$

using Fubini's Theorem. ■

Proof of Corollary 7. By Rio's covariance inequality,

$$Cov\left(g_Y\left(Y\right), g_Z\left(Z\right)\right) \leq 2 \int_0^{\alpha(g_Y, g_Z)} Q_{g_Y}\left(s\right) Q_{g_Z}\left(s\right) ds.$$

Since g_Y and g_Z are nondecreasing and right continuous,

$$\begin{aligned} & \sup_{z,y\in\mathbb{R}} \left|\Pr\left(g_Y\left(Y\right) > y, g_Z\left(Z\right) > z\right) - \Pr\left(g_Y\left(Y\right) > y\right)\Pr\left(g_Z\left(Z\right) > z\right)\right| \\ = & \sup_{z,y\in\mathbb{R}} \left|\Pr\left(Y > g_Y^{-1}\left(y\right), Z > g_Z^{-1}\left(z\right)\right) - \Pr\left(Y > g_Y^{-1}\left(y\right)\right)\Pr\left(Z > g_Z^{-1}\left(z\right)\right)\right|, \end{aligned}$$

so, by Theorem 4,

$$\alpha\left(g_Y, g_Z\right) = \alpha\left(Y, Z\right) \leq 2d_{M_1}\left(\mathbb{P}_{(Y,Z)}, \mathbb{P}_Y\mathbb{P}_X\right) \leq 2\min\left\{\left|8d_{BL_1}\left(\mathbb{P}_{(Y,Z)}, \mathbb{P}_Y\mathbb{P}_Z\right)\right|^{1/2}, 1\right\}.$$

■

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